Friday September 22nd, 2023, Time: 1 hour and 30 minutes

If you use a result proven in class then please state it clearly and verify the hypothesis while using the same.

1. (20 points) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space and X be any random variable. Assume the formula

$$\mathbb{E}[\mid X \mid] = \int_0^\infty \mathbb{P}(\mid X \mid \geq t) dt$$

   (a) Using the above formula evaluate the expectation when $X \sim \text{Exponential}(10)$

   (b) Suppose we know that $\mathbb{E}[\mid X \mid] < \infty$ and X, X_1, X_2, $\dots$ are i.i.d. random variables. Consider the n-th order statistic $X^n_{(n)} = \max\{X_1, \cdots, X_n\}$. Show

$$\mathbb{P}(\mid X^n_{(n)} \mid > t) \leq n\mathbb{P}(\mid X \mid > t)$$

   and

$$\frac{1}{n}\mathbb{E}(\mid X^n_{(n)} \mid) \to 0.$$

2. (20 points) Consider

$$\Omega = \left\{ \omega = (\omega_1, \omega_2, \dots,) : \omega_i \in \{0, 1\} \right\}$$

   equipped with probability $\mathbb{P}$ such that

$$\mathbb{P}(X_k = 0) = \mathbb{P}(X_k = 1) = \frac{1}{2}$$

   where for $1 \leq k$, $X_k : \Omega \to \{0, 1\}$ be given by $X_k(\omega) = \omega_k$. Suppose for $1 \leq n$, let $S_n : \Omega \to \mathbb{Z}$ be given by $S_n(\omega) = \sum_{k=1}^n X_k(\omega)$ and $S_0 = 0$. Show that for $a < \frac{1}{2}$

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \leq an) = \begin{cases} -\log(2) - a\log(a) - (1-a)\log(1-a) & \text{if } 0 < a < \frac{1}{2} \\ -\infty & \text{if } a < 0 \end{cases}$$

3. (20 points) Let $N \in \mathbb{N}$. Consider

$$\Omega_N = \left\{ \omega = (\omega_1, \omega_2, \dots, \omega_N) : \omega_i \in \{-1, +1\} \right\}$$

   equipped with the uniform distribution, denoted by $\mathbb{P} \equiv \mathbb{P}_N$.
   For $1 \leq k \leq N$, let $X_K : \Omega_N \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and
   for $1 \leq n \leq N$, let $S_n : \Omega_N \to \mathbb{Z}$ be given by $S_n(\omega) = \sum_{k=1}^n X_k(\omega)$ and $S_0 = 0$.

   (a) For $1 \leq n \leq N$, show that the mode of S_n is $\{0, 1\}$ that is

$$\max\{\mathbb{P}(S_n = a) : a \in \mathbb{Z}\} = \begin{cases} \mathbb{P}(S_{2k} = 0) & \text{if } n = 2k, k \in \mathbb{N} \\ \mathbb{P}(S_{2k-1} = 1) & \text{if } n = 2k-1, k \in \mathbb{N} \end{cases} = \binom{2k}{k}\frac{1}{2^{2k}}$$

   (b) For $a < b, a, b \in \mathbb{Z}$, $1 \leq n \leq N$ show that

$$\mathbb{P}(a \leq S_n \leq b) \leq (b - a + 1)\mathbb{P}(S_n \in \{0, 1\})$$

   and conclude that $\lim_{N\to\infty} \mathbb{P}(a \leq S_N \leq b) = 0$.

   (c) Let $\tau_N = \min\{\sigma_a, \sigma_b, N\}$. Show that τ_N is a Stopping time.